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LETTER TO THE EDITOR

The Schrödinger equation in terms of probability densities

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Abstract. The probability density of a non-relativistic particle is written as the sum of two non-negative density functions. The Schrödinger equation is equivalent to continuity equations (with source terms) for these functions. The equations are interpreted as describing two modes of the particle, which continually transform into each other.

The standard interpretation of the Schrödinger wavefunction ψ, given by Born in 1926, is that the function $P = \psi^*\psi$ is the probability density for the position of the particle [1]. In the same year, Madelung showed that the Schrödinger equation is equivalent to the continuity equation for P and another equation, which he identified as the Hamilton–Jacobi equation of the particle [2]. Madelung's identification has not been widely accepted, although it was the basis of Bohm's interpretation of quantum mechanics [3, 4].

We are going to show that the Schrödinger equation is equivalent to two continuity equations with source terms. This is simpler than Madelung's decomposition, and has a more compelling physical interpretation.

The Schrödinger equation for a particle of mass m in a potential U is

$$-(\hbar^2/2m)\nabla^2\psi + mU\psi - \mathrm{i}\hbar\partial_t\psi = 0 \tag{1}$$

where t is the time coordinate and $\partial_t = \partial/\partial t$. We write $\psi = \psi_R + \mathrm{i}\psi_I$, where ψ_R and ψ_I are real, and find that

$$\begin{aligned} -(\hbar^2/2m)\nabla^2\psi_R + mU\psi_R + \hbar\partial_t\psi_I &= 0 \\ -(\hbar^2/2m)\nabla^2\psi_I + mU\psi_I - \hbar\partial_t\psi_R &= 0. \end{aligned} \tag{2}$$

Velocities $\boldsymbol{u}$ and $\boldsymbol{v}$ are defined by

$$\nabla\psi_R = -m\hbar^{-1}\psi_I\boldsymbol{u} \qquad \nabla\psi_I = m\hbar^{-1}\psi_R\boldsymbol{v}. \tag{3}$$

Since $\psi_I\nabla^2\psi_R = \mathrm{div}(\psi_I\nabla\psi_R) - \nabla\psi_R\cdot\nabla\psi_I$ etc, equations (2) imply that

$$\mathrm{div}(P_R\boldsymbol{v}) + \partial_t P_R = \sigma \qquad \mathrm{div}(P_I\boldsymbol{u}) + \partial_t P_I = -\sigma \tag{4}$$

where $P_R = \psi_R^2$, $P_I = \psi_I^2$ and the source term σ is given by

$$\sigma = m\hbar^{-1}\psi_R\psi_I(2U - \boldsymbol{u}\cdot\boldsymbol{v}). \tag{5}$$

We note that σ does not involve the derivatives of $\boldsymbol{u}$ or $\boldsymbol{v}$.

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The definitions (3) were chosen to give the continuity-type equations (4). Although u or v may be singular when ψ_I or ψ_R vanishes, equations (4) are saved from singularity by the multiplying factors ψ_I and ψ_R. Equations (4) imply (1) at all points where ψ_I and ψ_R do not vanish.

If we write $P = P_R + P_I = \psi^*\psi$ and define s by $P_R v + P_I u = Ps$ equations (4) imply that

$$\operatorname{div}(Ps) + \partial_t P = 0 \tag{6}$$

which is the equation of continuity for the probability density P. (If one writes $\psi = \mathrm{e}^{(T+\mathrm{i}S)/\hbar}$, where T and S are real, it is easy to show that $\nabla S = ms$, and that $\nabla T = mw$, where $Pw = \psi_R\psi_I(v-u)$.)

To understand the significance of (4), we integrate over a volume V of 3 space and use the divergence theorem:

$$(\mathrm{d}/\mathrm{d}t)\int_V P_R\,\mathrm{d}^3x = -\int_{\partial V} P_R v\cdot \mathbf{n}\,\mathrm{d}A + \int_V \sigma\,\mathrm{d}^3x \tag{7}$$

$$(\mathrm{d}/\mathrm{d}t)\int_V P_I\,\mathrm{d}^3x = -\int_{\partial V} P_I u\cdot n\,\mathrm{d}A - \int_V \sigma\,\mathrm{d}^3x \tag{8}$$

where ∂V is the surface of V and n is the unit outward normal. To interpret this, we regard the particle as having two modes, R and I. The left-hand side of (7) is the rate of change of the probability that the particle in V is in mode R. The first term on the right-hand side is the rate of change of that probability due to the flux of particles in mode R across the surface of V; the second term represents the creation of particles in mode R inside V. Equation (8) has an exactly similar interpretation for particles in mode I. Because the last term in (8) is the negative of that in (7), the rate at which particles in mode I are destroyed is the same as the rate at which particles in mode R are created. In other words, σ is the rate of transformation (per unit time and per unit volume) from the I mode to the R mode.

It is now easy to visualize the meaning of equations (4). One can picture the two modes as corresponding to fluids, with densities P_R and P_I and velocity fields v and u, that transform into each other while conserving their total density. We must not, of course, take this classical picture too seriously: one cannot eliminate typically quantum effects, such as interference, by a mere change of formalism. If, for example, ψ' and ψ'' are solutions of (1), then so is $\psi = \psi' + \psi''$. The corresponding density functions are (P'_R, P'_I), (P''_R, P''_I), and (P_R, P_I), where $P_R = (\psi'_R + \psi''_R)^2 = P'_R + P''_R + 2\psi'_R\psi''_R$ and $P_I = (\psi'_I + \psi''_I)^2 = P'_I + P''_I + 2\psi'_I\psi''_I$, and the last terms represent the interference effects. It follows from (3) that $\psi_I u = \psi'_I u' + \psi''_I u''$, $\psi_R v = \psi'_R v' + \psi''_R v''$, and hence that $P_I u^2 = P'_I u'^2 + P''_I u''^2 + 2\psi'_I\psi''_I u'\cdot u''$ and $P_R v^2 = P'_R v'^2 + P''_R v''^2 + 2\psi'_R\psi''_R v'\cdot v''$. Again the interference effects vanish if $\psi'_R\psi''_R = 0$ and $\psi'_I\psi''_I = 0$.

The R and I modes are separately invariant under time reversal (they do not transform into each other like particles and antiparticles in quantum field theory). Time reversal is the transformation $t \mapsto t' = -t$, $\psi(x,t) \mapsto \psi'(x,t') = \psi^*(x,-t)$, $U(x,t) \mapsto U'(x,t') = U(x,-t)$; it leaves the Schrödinger equation invariant. We define ψ'_I, ψ'_R, u', v', etc, in an exactly similar way to ψ_I, ψ_R, u, and v, etc. Dropping the arguments x, t, and t', we get $\psi'_I = -\psi_I$, $\psi'_R = \psi_R$, $P'_I = P_I$, $P'_R = P_R$, $u' = -u$, $v' = -v$, $\sigma' = -\sigma$. Each of equations (4) is therefore invariant, and the R and I modes are unchanged by time reversal.

It may be accidental, a mere mathematical curiosity, that the Schrödinger equation can be written in the form (4). On the other hand, if (4) do describe two modes of a particle, we have gained a new insight into quantum mechanics. This new idea could be useful in the interpretation of quantum theory and the theory of measurement.

References

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